Problem. (Section 8.5 #31)

$$\int \frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2} d\theta$$

Solution. We do the partial fraction and integral in two steps.

(1) Partial fraction

It is highly advisable if you check that the denominator is **actually** irreducible. **The way to check is the sign of quadratic formula discriminant**, i.e.

$$b^2-4ac>0 \text{ (reducible, i.e. has real roots)} \quad \text{or} \quad <0 \text{ (irreducible, i.e. does not have real roots)?}$$

where a, b, c are from the form

$$a\theta^2+b\theta+c$$

Indeed, here $a=1, b=2, c=2$ and thus,

$$b^2-4ac=2^2-4\times 1\times 2=-4<0$$

which shows that the numerator is irreducible. Therefore, the numerator of the partial fraction must be of the form

$$A\theta+B$$

At the same time, this factor is also **repeated**. One then must include powers of the denominator up to the **multiplicity** of the repetition. Here, the multiplicity is 2.

Combining the argument above, the partial fraction now reads, for unknowns A, B, C, D,

$$\frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2}=\frac{A\theta+B}{\theta^2+2\theta+2}+\frac{C\theta+D}{\left(\theta^2+2\theta+2\right)^2}$$

Combining the fractions on the right hand side, we have

$$\frac{(A\theta+B)\left(\theta^2+2\theta+2\right)+C\theta+D}{\left(\theta^2+2\theta+2\right)^2}=\frac{A\theta^3+B\theta^2+2A\theta^2+2B\theta+2A\theta+2B+C\theta+D}{\left(\theta^2+2\theta+2\right)^2}$$

$$\frac{A\theta^3+(B+2A)\theta^2+(2B+2A+C)\theta+(2B+D)}{\left(\theta^2+2\theta+2\right)^2}$$

where I have arranged the numerator in **descending powers of** θ. It remains to match (solve) the following equation,

$$2\theta^3+5\theta^2+8\theta+4=A\theta^3+(B+2A)\theta^2+(2B+2A+C)\theta+(2B+D)$$

There is only one cubic term, and thus

$$A=2$$

The quadratic term yields

$$5=B+2A \implies B=1$$

The linear term yields

$$8=2B+2A+C \implies C=8-2B-2A=2$$

and the constant term yields

$$4=2B+D \implies D=4-2B=2$$

The onion peeling is finished. Complete partial fraction gives

$$\frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2}=\frac{2\theta+1}{\theta^2+2\theta+2}+\frac{2\theta+2}{\left(\theta^2+2\theta+2\right)^2}$$

(2) Integral

$$\int \frac{2\theta^3+5\theta^2+8\theta+4}{\left(\theta^2+2\theta+2\right)^2} d\theta=\boxed{\int \frac{2\theta+1}{\theta^2+2\theta+2} d\theta+\int \frac{2\theta+2}{\left(\theta^2+2\theta+2\right)^2} d\theta}$$

The **second integral** is easier because with a **substitution** of

$$u=\theta^2+2\theta+2$$

the derivative is readily in the numerator,

$$du = (2\theta + 2)\, d\theta$$

thus reducing the second integral into

$$\int \frac{2\theta + 2}{\left(\theta^2 + 2\theta + 2\right)^2} d\theta = \int \frac{1}{u^2} dx = -\frac{1}{u} = \boxed{-\frac{1}{\theta^2 + 2\theta + 2}}.$$

The **first integral** requires more thoughts. We first rewrite the numerator as $2\theta + 2 - 1$ and split the integral,

$$\begin{aligned}
\int \frac{2\theta + 1}{\theta^2 + 2\theta + 2} d\theta &= \int \frac{2\theta + 2 - 1}{\theta^2 + 2\theta + 2} d\theta \\
&= \int \frac{2\theta + 2}{\theta^2 + 2\theta + 2} d\theta - \int \frac{1}{\theta^2 + 2\theta + 2} d\theta \\
&\overset{\dagger}{=} \int \frac{1}{u} du - \int \frac{1}{\theta^2 + 2\theta + 2} d\theta \\
&= \boxed{\ln\left(\theta^2 + 2\theta + 2\right) - \int \frac{1}{\theta^2 + 2\theta + 2} d\theta}
\end{aligned}$$

where we used again the substitution $u = \theta^2 + 2\theta + 2$ at $\dagger$. The second integral requires completing the square in the denominator, namely,

$$\begin{aligned}
\int \frac{1}{\theta^2 + 2\theta + 2} d\theta &= \int \frac{1}{\left(\theta^2 + 2\theta + 1\right) + 1} d\theta \\
&= \int \frac{1}{\left(\theta + 1\right)^2 + 1} d\theta \\
&\overset{w=\theta+1}{=} \int \frac{1}{w^2 + 1} dw \\
&= \tan^{-1}(w) \\
&= \boxed{\tan^{-1}(\theta + 1)}
\end{aligned}$$

Linking everything in the boxes above, we arrive at

$$\boxed{\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{\left(\theta^2 + 2\theta + 2\right)^2} d\theta = \ln\left(\theta^2 + 2\theta + 2\right) - \tan^{-1}(\theta + 1) - \frac{1}{\theta^2 + 2\theta + 2} + C}$$